Physics 523, General Relativity

Homework 6

Due Wednesday, 29th November 2006

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Problem 1[1]

We are asked to determine the ratio of frequencies observed at two fixed[2] points in a spacetime with a static metric g_{ab}; we should use this to determine the redshift of light emitted from the surface of the Sun which is observed on the surface of the Earth.

Imagine a clock at a fixed point x_1 which ticks with a regular interval Δs. Because the point is stationary, we may use the definition of the spacetime metric g_{ab} to see that this interval is related to the coordinate time interval Δt by[3]

$$\Delta s^2 = \Delta t_1^2 g_{00}(x_1). \tag{1.1}$$

We have included a subscript on the coordinate time interval to make its position-dependence manifest. The invariant interval Δs, however, must certainly be position-independent for any reliable clock. Therefore, we naturally have that

$$\Delta s^2 = \Delta t_1^2 g_{00}(x_1) = \Delta t_2^2 g_{00}(x_2), \tag{1.2}$$

for any other point x_2. This implies that

$$\therefore \frac{\Delta t_1^2}{\Delta t_2^2} = \frac{g_{00}(x_2)}{g_{00}(x_1)}. \tag{1.3}$$

It is important to note that this discussion is not limited to clocks ticking regularly: any process with a well-defined, constant time interval observed at two distinct points will obey equation (1.3). Indeed, consider an atomic transition which emits photons with frequency $\nu_1 \equiv \frac{1}{\Delta t_1}$ at point x_1. Equation (1.3) implies that the frequency at x_1 will be related to the frequency ν_2 at x_2 by

$$\therefore \frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}}. \tag{1.4}$$

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To determine the redshift of light emitted from the Sun and observed on the Earth we recall that in the Newtonian (weak-field) approximation,

$$g_{00}(x) = -1 - 2\varphi_N(x), \tag{1.5}$$

where $\varphi_N(x)$ is the Newtonian potential at x. The only subtlety is that we should make sure to be careful about units when computing $\varphi_N(x)$. Notice that because the '1' in $-1-2\varphi_N(x)$ is dimensionless, so should $\varphi_N(x)$ be. This will be the case if we judiciously set $c = 1$. In these units, we find

$$\varphi_N(R_\odot) = -2.12 \times 10^{-6} \qquad \text{and} \qquad \varphi_N(R_\oplus) = -1.06 \times 10^{-8}, \tag{1.6}$$

which gives a redshift of 2.11 parts per million.

[1] Note added in revision: this solution is bad. The argument presented for equation (1.4) is not valid (even though the right answer emerges). One should be very careful about the thought experiment under consideration (because the inverse result is easy to obtain under a different situation).

[2] The equation which the problem set asks us to demonstrate is only valid for *stationary* sources and observers—otherwise there would be a doppler-shift term obfuscating the equation.

[3] In his textbook, Weinberg has an interesting discussion on why it is fundamentally not possible to disentangle Δs from Δt at a particular point. However, it *is* possible to compare the metric at two distinct points—by observing a gravitational redshift—as described presently.

Problem 2

We are to find the 'natural' generally covariant generalization of the flat-space Klein-Gordon Lagrangian (which was shown to be Weyl invariant in the last problem set). We should use this to determine the matter stress-energy tensor and show that it is traceless.

The striking similarity between

$$g^{ab}\nabla_a\nabla_b\varphi - \frac{1}{6}R\varphi = 0, \tag{2.1}$$

and the massive Klein-Gordon equation makes us guess that the action from which this is derived is

$$S = \frac{1}{2}\int d^4x\sqrt{-g}g^{ab}\left(\nabla_a\varphi\nabla_b\varphi + \frac{1}{6}R_{ab}\varphi^2\right). \tag{2.2}$$

Our intuition is confirmed by calculating the equation of motion:

$$\begin{aligned} 0 &= \nabla_a\left(\frac{\partial\mathscr{L}}{\partial\nabla_a\varphi}\right) - \frac{\partial\mathscr{L}}{\partial\varphi} = \nabla_a\left(g^{ab}\nabla_b\varphi\right) - \frac{1}{6}R\varphi, \\ &= g^{ab}\nabla_a\nabla_b\varphi - \frac{1}{6}R\varphi. \end{aligned} \tag{2.3}$$

Therefore the action (2.2) does indeed give rise to the desired equation of motion for φ as desired.

We now must compute the stress-energy tensor for this matter Lagrangian. Recall that the stress-energy tensor T^{ab} of a system with action S is defined according to

$$\delta S = \frac{1}{2}\int d^4x\sqrt{-g}\; T^{ab}\delta g_{ab}, \tag{2.4}$$

from the variation $g_{ab} \mapsto g_{ab} + \delta g_{ab}$. To compute the metric variation for the action given in (2.2) we first recall some useful identities:

$$\delta g^{ab} = -g^{ac}g^{bd}\delta g_{cd}; \qquad \delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}; \tag{2.5}$$

$$\text{and} \qquad g^{ab}\delta R_{ab} = \nabla^a w_a, \qquad \text{where} \qquad w_a \equiv \nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right). \tag{2.6}$$

This last identity, (2.6), follows from work done in lecture. Although brevity tempts us to simply quote Wald's textbook, it is sufficiently important to warrant a full derivation. Therefore, to please the reader, a proof of this identity has been included as an Appendix to this problem set.

We are now prepared to compute the metric variation of the action (2.2). As we proceed, any total divergence will be assumed to integrate to zero.

$$\begin{aligned} \delta S =& \frac{1}{2}\int d^4x\sqrt{-g}\left\{\frac{1}{2}g^{ab}\delta g_{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) + \delta g^{ab}\left(\nabla_a\varphi\nabla_b\varphi + \frac{1}{6}R_{ab}\varphi^2\right) + \frac{1}{6}g^{ab}\delta R_{ab}\varphi^2\right\}, \\ =& \frac{1}{2}\int d^4x\sqrt{-g}\left[\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) - \left(\nabla^a\varphi\nabla^b\varphi + \frac{1}{6}R^{ab}\varphi^2\right)\right\} + \frac{1}{6}g^{cd}\delta R_{cd}\varphi^2\right], \\ =& \frac{1}{2}\int d^4x\sqrt{-g}\left[\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) - \left(\nabla^a\varphi\nabla^b\varphi + \frac{1}{6}R^{ab}\varphi^2\right)\right\} + \frac{1}{6}\left(\nabla^c w_c\right)\varphi^2\right], \\ =& \frac{1}{2}\int d^4x\sqrt{-g}\left[\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) - \left(\nabla^a\varphi\nabla^b\varphi + \frac{1}{6}R^{ab}\varphi^2\right)\right\} - \frac{1}{6}\left(\nabla^c\varphi^2\right)w_c\right]. \end{aligned} \tag{2.7}$$

The last term in the expression above is qualitatively different from the first two. Let us try to recast it into a form which makes the δg_{ab}-dependence manifest. Using the definition of w_c and making repeated use of integration by parts, we see

$$\begin{aligned} \int d^4x\sqrt{-g}\;\nabla^a(\varphi^2)w_a &= \int d^4x\sqrt{-g}\;\nabla^a(\varphi^2)\Big(\nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right)\Big), \\ &= \int d^4x\sqrt{-g}\;\left(\nabla^b\Big(\delta g_{ab}\nabla^a(\varphi^2)\Big) - \delta g_{ab}\nabla^b\Big(\nabla^a(\varphi^2)\Big) - g^{cd}\nabla_a\Big(\delta g_{cd}\nabla^a(\varphi^2)\Big) + g^{ab}\delta g_{ab}\nabla_c\Big(\nabla^c(\varphi^2)\Big)\right), \\ &= \int d^4x\sqrt{-g}\;\delta g_{ab}\Big(g^{ab}g^{cd}\nabla_c\nabla_d\varphi^2 - \nabla^a\nabla^b\varphi^2\Big). \end{aligned} \tag{2.8}$$

We are now ready to put everything together and find T^{ab}. To make our result a bit more transparent, let us agree to call $g^{cd}\nabla_c\nabla_d \equiv \Box$. Also, the identity $g^{cd}\nabla_c\varphi\nabla_d\varphi = \frac{1}{2}\Box\varphi^2 - \varphi\Box\varphi$ will allow us to tidy up our expressions substantially. Combining all of this, we can continue our work on the total variation (2.7) using the result from (2.8) to find

$$\begin{aligned}
\delta S &= \frac{1}{2}\int d^4x\sqrt{-g}\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) - \left(\nabla^a\varphi\nabla^b\varphi + \frac{1}{6}R^{ab}\varphi^2\right) - \frac{1}{6}\left(g^{ab}g^{cd}\nabla_c\nabla_d\varphi^2 - \nabla^a\nabla^b\varphi^2\right)\right\},\\
&= \frac{1}{2}\int d^4x\sqrt{-g}\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi - \frac{1}{3}\nabla_c\nabla_d\varphi^2 + \frac{1}{6}R_{cd}\varphi^2\right) - \nabla^a\varphi\nabla^b\varphi - \frac{1}{6}R^{ab}\varphi^2 + \frac{1}{6}\nabla^a\nabla^b\varphi^2\right\},\\
&= \frac{1}{2}\int d^4x\sqrt{-g}\delta g_{ab}\left\{\frac{1}{2}g^{ab}\left(\frac{1}{6}\Box\varphi^2 + \frac{1}{6}R\varphi^2 - \varphi\Box\varphi\right) - \frac{1}{3}\nabla^a\nabla^b\varphi^2 + \varphi\nabla^a\nabla^b\varphi - \frac{1}{6}R^{ab}\varphi^2\right\}. \qquad (2.9)
\end{aligned}$$

This allows us to read-off

$$\therefore T^{ab} = \frac{1}{2}g^{ab}\left(\frac{1}{6}\Box\varphi^2 + \frac{1}{6}R\varphi^2 - \varphi\Box\varphi\right) - \frac{1}{3}\nabla^a\nabla^b\varphi^2 + \varphi\nabla^a\nabla^b\varphi - \frac{1}{6}R^{ab}\varphi^2. \qquad (2.10)$$

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As anyone who's seen conformal field theory knows, the trace of the stress-energy tensor must vanish. Let's see how this 'magically' works out in the situation considered presently.

$$\begin{aligned}
g_{ab}T^{ab} &= \frac{1}{3}\Box\varphi^2 + \frac{1}{3}R\varphi^2 - 2\varphi\Box\varphi - \frac{1}{3}\Box\varphi^2 + \varphi\Box\varphi - \frac{1}{6}R\varphi^2,\\
&= \frac{1}{6}R\varphi^2 - \varphi\Box\varphi,\\
&= -\varphi\left(\Box\varphi - \frac{1}{6}R\varphi\right),\\
&= 0.
\end{aligned}$$

Notice that the last line required using the equations of motion—which wasn't entirely anticipated—at least by us.

Problem 3: Killing Vectors

a. If $\zeta_a(x)$ is a Killing field and $p^a(\lambda)$ is the tangent vector to a geodesic curve $\gamma(\lambda)$, then $p^a\zeta_a(x)$ is constant along γ.

proof: The derivative of $p^a\zeta_a$ along γ is

$$p^b\nabla_b\left(p^a\zeta_a\right) = p^a p^b\nabla_b\zeta_a + \zeta_a p^b\nabla_b p^a. \tag{3.1}$$

The first term vanishes because $p^a p^b$ is symmetric while $\nabla_b\zeta_a$ is antisymmetric (because it is Killing). The second term vanishes because p^a is the tangent of a geodesic, which practically by definition implies that it obeys the geodesic equation, $p^b\nabla_b p^a = 0$.

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b. We are to list the ten independent Killing fields of Minkowski spacetime.

The ten independent Killing fields correspond to the ten generators of the Poincaré algebra: four translations, three rotations, and three boosts. Given in terms of the basis vectors $\vec{e}_a$, we the Killing *vector* fields are therefore

Translations :	$\vec{e}_t,$	$\vec{e}_x,$	$\vec{e}_y,$	$\vec{e}_z;$
Rotations :	$y\vec{e}_x - x\vec{e}_y,$	$z\vec{e}_y - y\vec{e}_z,$	$x\vec{e}_z - z\vec{e}_x;$	
Boosts :	$x\vec{e}_t + t\vec{e}_x,$	$y\vec{e}_t + t\vec{e}_y,$	$z\vec{e}_t + t\vec{e}_z;$	

Each of these ten vector fields manifestly satisfies Killing's equation. That they are linearly independent is also manifest[4].

c. If ζ_a and η_a are Killing fields and α, β constants, then $\alpha\zeta_a + \beta\eta_a$ is Killing.

proof: As should be obvious to all but the most casual observer,

$$\nabla_b\left(\alpha\zeta_a + \beta\eta_z\right) = \alpha\nabla_b\zeta_a + \beta\nabla_b\eta_a = -\alpha\nabla_a\zeta_b - \beta\nabla_a\eta_b = -\nabla_a\left(\alpha\zeta_b + \beta\eta_b\right), \tag{3.2}$$

because, being constants, α, β commute with the gradient and ζ_a, η_a are Killing. Therefore equation (3.2) implies that $(\alpha\zeta_a + \beta\eta_a)$ is Killing.

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d. We are to show that Lorentz transformations of the Killing vector fields listed in part (**b**) above give rise to linear recombinations of the same fields with constant coefficients.

Because every Lorentz transformation can be built from infinitesimal ones, it is sufficient to demonstrate the claim for infinitesimal Lorentz transformations. And this makes our work exceptionally easy. Infinitesimal Lorentz transformations are simply the identity plus a constant multiple of the generators of the Lorentz algebra; but (the last six of) the Killing fields listed in part (**b**) are nothing but these Lorentz generators.

Therefore, any infinitesimal Lorentz transformation of the Killing fields listed in part (**b**) is a linear combination of those same Killing fields with constant coefficients. And by extension, the same is true for any finite Lorentz transformation.

[4]Although we should add that we were not requested to demonstrate this—so our lack of exposition here should be forgiven.

Appendix

In problem 2 we made use of an identity that didn't obviously follow from the work in lecture. We remedy that deficiency presently[5].

Lemma: Under the variation $g_{ab} \mapsto g_{ab} + \delta g_{ab}$,

$$g^{ab}\delta R_{ab} = \nabla^a w_a, \qquad \text{where} \qquad w_a \equiv \nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right). \tag{A.1}$$

proof: We may begin with the related expression derived in lecture,

$$g^{ab}\delta R_{ab} = \nabla_a\Big(g^{bc}\delta\Gamma^a_{bc} - g^{ac}\delta\Gamma^b_{cb}\Big). \tag{A.2}$$

Rearranging this we find

$$g^{ab}\delta R_{ab} = \nabla^a\Big\{g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad}\Big\}; \tag{A.3}$$

therefore, it suffices to show that the term in brackets is equal to w_a. Expanding this expression and using symmetry to collect and cancel terms, we find

$$\begin{aligned}
g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad} =& \frac{1}{2}g^{bc}g_{ad}\delta g^{de}\Big(g_{be,c} + g_{ce,b} - g_{bc,e}\Big) + \frac{1}{2}g^{bc}\delta^e_a\Big(\delta g_{be,c} + \delta g_{ce,b} - \delta g_{bc,e}\Big)\\
&- \frac{1}{2}\delta g^{be}\Big(g_{ae,b} + g_{be,a} - g_{ab,e}\Big) - \frac{1}{2}g^{be}\Big(\delta g_{ae,b} + \delta g_{be,a} - \delta g_{ab,e}\Big),\\
=& -\frac{1}{2}g^{bc}g^{ef}\delta^d_a\delta g_{de}\Big(g_{bf,c} + g_{cf,b} - g_{bc,f}\Big) + \frac{1}{2}g^{bc}\Big(\delta g_{ba,c} + \delta g_{ca,b} - \delta g_{bc,a}\Big) + \frac{1}{2}\delta g_{de}g^{dc}g^{ef}g_{cf,a} - \frac{1}{2}g^{be}\delta g_{be,a},\\
=& -g^{bc}g^{ef}\delta^d_a\delta g_{de}g_{bf,c} - \frac{1}{2}g^{bc}g^{ef}\delta^d_a\delta g_{de}g_{bc,f} + g^{bc}\delta g_{ba,c} - \frac{1}{2}g^{bc}\delta g_{bc,a} + \frac{1}{2}\delta g_{de}g^{dc}g^{ef}g_{cf,a} - \frac{1}{2}g^{bc}\delta g_{bc,a},\\
=& g^{bc}\delta g_{ba,c} - g^{bc}\delta g_{bc,a} - g^{bc}g^{ef}g_{bf,c}\delta g_{ae} - \frac{1}{2}g^{bc}g^{ef}g_{bc,f}\delta g_{ae} + \frac{1}{2}g^{dc}g^{ef}g_{cf,a}\delta g_{de}.
\end{aligned}$$

Expanding the first two terms in the expression above in terms of covariant derivatives and Christoffel symbols, we observe

$$g^{bc}\delta g_{ab,c} = \nabla^b\left(\delta g_{ab}\right) + g^{bc}\delta g_{eb}\Gamma^e_{ac} + g^{bc}\delta g_{ae}\Gamma^e_{bc}, \tag{A.4}$$

and

$$g^{bc}\delta g_{bc,a} = g^{bc}\nabla_a\left(\delta g_{bc}\right) + g^{bc}\delta g_{be}\Gamma^e_{ac} + g^{bc}\delta g_{ec}\Gamma^e_{ba}. \tag{A.5}$$

Noting that the terms with the covariant derivatives are what we are looking for—together, they give w_a. Putting everything together,

$$g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad} = w_a + \left\{g^{bc}\delta g_{ae}\Gamma^e_{bc} - g^{bc}\delta g_{ec}\Gamma^e_{ba} + g^{bc}g^{ef}\left(\frac{1}{2}g_{fc,a}\delta g_{be} + \frac{1}{2}g_{bc,f}\delta g_{ae} - g_{bf,c}\delta g_{ae}\right)\right\}. \tag{A.6}$$

All that remains is for us to show that the terms in curly brackets above vanish. To do this, we will expand our expressions one last time—this time using the definition of the Christoffel symbols for a metric connection. Doing so, we find

$$\begin{aligned}
g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad} - w_a = g^{bc}g^{ef}\Big\{& \frac{1}{2}g_{fc,a}\delta g_{be} + \frac{1}{2}g_{bc,f}\delta g_{ae} - g_{bf,c}\delta g_{ae}\\
&- \frac{1}{2}g_{bf,a}\delta g_{ec} - \frac{1}{2}g_{af,b}\delta g_{ec} + \frac{1}{2}g_{ba,f}\delta g_{ec}\\
&+ \frac{1}{2}g_{bf,c}\delta g_{ae} + \frac{1}{2}g_{cf,b}\delta g_{ae} - \frac{1}{2}g_{bc,f}\delta g_{ae}\Big\},\\
=& 0.
\end{aligned}$$

Here we have indicated the terms that cancel together in matching colours. With this, we have shown that

$$\therefore g^{ab}\delta R_{ab} = \nabla^a\Big\{g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad}\Big\} = \nabla^a\Big\{\nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right)\Big\} = \nabla^a w_a. \tag{A.7}$$

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[5]We hope that there is an easier way to prove the following Lemma. But alas! too little time to be brief. Breviloquence is a time-consuming luxury.